Realizing the Quantum Relative Entropy of Two Noisy States using the Hudson-Parthasarathy Equations

Bhaveshkumar B. Prajapati[1] and Nirbhay Kumar Chaubey[2]

Asst. Professor, IT Department, L.D. College of Engineering & Research Scholar, GTU, Gujarat, India[1]

Senior Member, IEEE

Professor and Dean of Computer Science, Ganpat University, Gujarat, India[2]

***Abstract*—The idea of noisy states can be derived through a quantum relative entropy over a given time period and construct the average value of X at time based on the system variables. A random Hermitian matrix is used to represent the quantum system observables with BATH states. The Hudson-Parthasarathy (HP equation) context for stochastic processes allows us to simulate quantum relative entropy using quantum Brownian motion. The Sudarshan-Lindblad's density evolution matrix equation was already derivable in generalized form in my previous work. This paper's goal is to illustrate how the HP equation may be used to estimate the density matrix for noise in a perturbed quantum system of a stochastic process. The last stage involves using MATLAB to estimate and simulate a random density matrix and measure the quantum average $T_r(\rho(t)X)$ at various times. These formulas would be helpful in determining how sensitive the evolving/evolved states are to changes in the Hamiltonian of the noise operators in a sensitivity/robustness study of quantum systems.**

Keywords*—Schrödinger equation; Ito calculus; quantum relative entropy; Hudson-Parthasarathy equation; quantum noise

I. INTRODUCTION

The origin of Quantum mechanics in 1925, has been expressed to solve many problems and conceived as a generalization of classical mechanics with an added quantum indeterminism [1]. A number of open problems in quantum information theory revolve around whether certain quantities are additive or not. The oldest one was the Holevo capacity method. According to this conjecture, entangled signal states do not improve quantum channel capacity. A second additivity conjecture concerns the minimum entropy of the quantum channel's output [2-4]. As a result, the linear Hilbert space structure is given priority while the probabilistic structure is added almost as an afterthought. It has the unfortunate consequence that in the standard approach to Quantum Mechanics, the dynamical and probabilistic aspects of quantum theory are not quite compatible. There are two distinct modes of wave function evolution: the linear Schrödinger evolution and the probabilistic wave function collapse [5].

A brief but interesting discussion is given on the computation of atomic transition probabilities when the atom interacts with an electromagnetic field. We then calculate using this expression of the atomic state, the average value of an observable on the quantum system as a function of time in terms of the information bearing sequence and use these formulae to derive estimates of the information sequence from a continuous measurement of the observable average [6-9].

After this, we obtain a more accurate description of the measurement and estimation process. When the quantum system is in a pure or mixed state, the measurement of an observable causes the state of the system to collapse to one whose range is contained in the orthogonal eigen-projection of the observable associated with the eigenvalue of X that has been observed as the outcome. After such a measurement at time t_1, the system again evolves from the collapsed state under the same Hamiltonian upto time $t_2 > t_1$ when once again the same observable is measured. Again the state of the system collapses to a state decided by the corresponding eigen-projection and the system evolves from this collapsed state. If P is the eigen-projection and ρ is the state just prior to the measurement, then the probability of observing the corresponding outcome is $T_r(\rho P)$. In this way, we are able to compute the joint probability of measuring a subset (possibly repeated) of eigenvalues of the observable at a finite set of times $t_1, \dots, t_N$, with each time the measurement being made, the system collapsing to a state corresponding to the associated eigen-projection [9-13]. Using quantum measurement models, we examine what kind of measurements can be made on quantum systems, as well as how to determine the probability that a measurement will yield a certain result. In order to find out the effective measurement of quantum states, that technique is very important because there should be a minimum uncertainty of the state with sensitivity to their environment.

One way to describe the output of a single mode, stabilized laser is as a coherent state. An analogy between quantum-mechanical and classical particles oscillating in a harmonic potential is coherent states [5], [7], [11] [24]. Coherent state are eigenstates of the annihilation operator. Coherent states are eigenstates of the annihilation operators' fields. In quantum field theory, creation and annihilation operators' fields are used to correlative the electromagnetic four potential vector field. Thus in a coherent state, one part of the electromagnetic field has defined amplitude and phase. The state $|n_1, n_2, \dots, n_N >$ correspond to n_j photons or the j^{th} type being present in the bath. Thus $|n_1, n_2, \dots, n_N >$ is an eigenstates of the number operators $a_k^\dagger a_k$ with eigenvalue n_k [8], [10], [19-23].

A bit flip is the only possible error in classical computing while bits are being transferred. Since any rotation or phase shift in Hilbert space represents an error, there are an endless number of distinct faults that could happen for a single qubit in the quantum scenario. During the measurement, a compatible subspace is projected from the quantum state. When the error is measured, it is brought down to a level that is reasonable for the measurement. In comparison to quantum noise, classical noise has fewer degrees of freedom, making it commutative unitary quantum noise. Classical noise can be created as a particular instance of quantum noise by taking into account the approximate states of both. The novelties of this approach are;

- The time-dependent creation, annihilation, and conservation are introduced in the unique method for quantum stochastic calculus developed by Hudson and Parthasarathy (HP), which satisfies the quantum Ito calculus formula for the product of time differentials of these processes.
- It is demonstrated that the quantum Ito formula of HP naturally evolves into a spectral commutative version of the classical Ito formula for Brownian motion and the Poisson process.
- The Boson Fock space, which is a family of non-commuting operators that specialize to Brownian motion when the state is selected appropriately [12–15], is shown to provide the basis for creating fundamental quantum noise processes in this paper [10–11].
- The linear stochastic model is generalized by HP equations.
- The Schrödinger equation defines a system's unitary evolution when it is coupled to a noisy environment. Because particles can move from the system into the bath and from the bath back into the system, total probability is conserved, which explains why system tensors with BATH exhibit joint unitary evolution [25–29].
- In the HP theory, quantum noise is just a family of non-random operators in Fock Space. When we examine the stochastic linear operator in particular states, randomness appears in all situations. The quantum theory naturally incorporates randomness [14–19]. The classical Ito table is generalized by the quantum Ito tables.

As a result, our main contribution to this study is that infinite-dimensional systems, such as the HP equation, must be truncated in order to achieve a finite-dimensional approximation, which can then be easily reduced utilizing MATLB through discrimination approaches. We have determined how quickly the respective entropies of the two quantum systems change. Based on their geometric measure of entanglement, some mixed states should allow for the analytical calculation of the rate of change of quantum relative Von Neumann entropy. The principle can be regarded of as a generalization of both the maximum entropy principle and the minimal entropy production principle, both of which are frequently employed in non-equilibrium thermodynamics. This justifies the employment of the principle in the context of optimum learning systems [30-34]. With the use of the symmetric tensor product of a specific Hilbert space, we create the Boson Fock space, which can explain any number of bosons. The Boson Fock space serves as the foundation for creating fundamental quantum noise processes, such as the noncommuting family of operators that, given the right state selection, specialize to Brownian motion and Poisson processes. A tensor product connects the system Hilbert space to the Boson Fock space, also known as the noise Bath space. Then, we construct the creation, annihilation, and conservation operator fields in the Boson Fock space in accordance with R.L. Hudson and K.R. Parthasarathy's wonderful methodology.

The rest of this paper is written as follows: In Section II, Observable of Quantum Systems using mathematical representations is described. In Section III, the mathematical model of quantum relative entropy for the evolution of two quantum systems is described. In Section IV, the NSER (noise-to-signal energy ratio) to validate the performance criterion is computed. Concluding thoughts are discussed in Section V.

II. Observable of Quantum Systems

A finite level of a quantum system $\{A, B\}$ and each system can be described by a finite dimension of Hilbert space $\{H^A, H^B\}$. An element of Hilbert space H is an $n \times n$ Hermitian matrix with complex entries, called ket vector $|u>$and if the same function is linear of the Hilbert space then it is bra vector$< v|$. The density matrix of a quantum mechanical system is used to compute the mean value of observables. An operator on a Hilbert space with unit trace that is positive semidefinite is called a density operator ρ. In order for an operator to be considered positive semi definite, it must be Hermitian and have no negative (necessarily real) eigenvalues [5], [22]. Let ρ is a density matrix of a quantum system and X, Y two observable on the same Hilbert space. Assume that, $T_r(\rho X) = T_r(\rho Y) = 0$. Note that $T_r(\rho[X, Y])$ is a purely imaginary complex number. A system observable is changed into a system plus noise variable after a finite amount of time by this unitary evolution, which operates on the tensor product of the system and noise Hilbert space. Based on observations made up to time t, an estimate of this noisy observable at each time t is required. In order to do this, the measurement process must, however, satisfy the non-demolition property, which requires that the measurement Von Neumann algebra is Abelian and that the measurement at time s commutes with the state at time t for time t≥s.

Suppose the observable X evolves in time as $X(t) = e^{itH}.X.e^{-itH}, t \geq 0$. Then, X(t) satisfies the Heisenberg equation of motion for observables:

$$\frac{dX(t)}{dt} = i[H, X(t)].$$

Let ρ be a density matrix on the same Hilbert space then $T_r(\rho(t)X = T_r(\rho X(t))$. For all observables X, then $\rho(t) = e^{-itH}.\rho.e^{itH}$ and deduce that,

$$\frac{d\rho(t)}{dt} = i[H, \rho(t)].$$

Therefore, these results can be interpreted in terms of Schrödinger's wave mechanics and Heisenberg's matrix mechanics. Let ρ_A and ρ_B be two destiny matrices on $\mathbb{C}^d$ (both are positive define with trace one). So, to determine a unitary matrix U such that $\| \rho_B - U\rho_A U^* \|$ is a minimum, where $\|.\|$ denotes Frobenius Norm [23].

Let U be the optimal unitary matrix. Then for any Hermitian matrix H must have,

$$\frac{d}{dt} \parallel \rho_B - Ue^{itH}\rho^A e^{-itH}U^* \parallel_{t=0}^2 = 0$$

This gives,

$$T_r((\rho_B - U\rho_A U^*)U[H, \rho_A]U^*) = 0$$

or equivalently,

$$T_r((U^*\rho_B U - \rho_A)[H, \rho_A]) = 0$$

or

$$T_r([\rho_A, U^*\rho_B U]H) = 0$$

For all Hermitian matrices H. It follows that U must satisfy $[\rho_A, U^*\rho_B U] = 0$ or $[U^*\rho_A U, \rho_B] = 0$. By performing an average over the bath noise variables at each time, we are able to describe how system observables evolve when they are corrupted by bath noise in a way that ensures the system observable always remains a system observable.

III. Mathematical Model of Quantum Relative Entropy

A quantum relative entropy is evolving in between two quantum systems $\rho_A(t)$ and $\rho_B(t)$ are density matrices satisfying the Sudarshan- Lindblad equation [33]:

$$\rho_A'(t) = -\iota[H_A, \rho_A(t)] - \frac{1}{2}\theta_1(\rho_A(t))$$

$$\rho_B'(t) = -\iota[H_B, \rho_B(t)] - \frac{1}{2}\theta_2(\rho_B(t))$$

Where, $\theta_1(X) = \sum_{k=1}^{p}(L_k^*L_kX + XL_k^*L_k - 2L_kXL_k^*)$

$$\theta_2(X) = \sum_{k=1}^{p}(M_k^*M_kX + XM_k^*M_k - 2M_k^*M_k)$$

Assume $H_2 - H_1$ and $M_k - L_k$ upto $O(\varepsilon)$, then calculate upto $O(\varepsilon^2)$.

$$\frac{d}{dt}T_r(\rho_A\log\rho_A) = T_r(\frac{d\rho_A}{dt}) + T_r(\rho_A\frac{d}{dt}\log\rho_A)$$

so by $\rho_A = e^{Z_1}, \rho_B = e^{Z_1}$,

$$\rho_A' = e^{Z_1}\frac{I - e^{-adZ_1}}{adZ_1}(Z_1')$$

Thus,

$$Z_1' = \rho_A^{-1}\sum_{r=1}^{\infty}c_r(adZ_1)^r(\rho_A^{-1}\rho_A')$$

$$T_r(\rho_A\frac{d}{dt}\log\rho_A) = T_r(\rho_A Z_1)$$

$$= \sum_{r=1}^{\infty}c_r T_r(\rho_A(ad\log\rho_A)^r(\rho_A^{-1}\rho_A'))$$

(since $T_r(\rho_A') = 0$), so

$$\frac{d}{dt}T_r(\rho_A\log\rho_A) = T_r[\rho_A'\log\rho_A] \text{ and,}$$

$$\frac{d}{dt}T_r(\rho_A\log\rho_B) = T_r(\rho_A'\log\rho_B) + T_(\rho_A Z_2')$$

$$Z_2 = \log\rho_B, \text{ then } Z_2' = \frac{adZ_2}{1-e^{-adZ_2}}(\rho_A^{-1}\rho_B')$$

$$T_r(\rho_A Z_2') = T_r\{\rho_A(adZ_2\sum_{m=0}^{\infty}e^{-m.adZ_2}(\rho_B^{-1}\rho_B'))\}$$

$$= \sum_{m=0}^{\infty}T_r(\rho_B^m\rho_A\rho_2^{-m-1}[Z_2, \rho_B'])$$

so, $\frac{d}{dt}S(\rho_A, \rho_B) = \frac{d}{dt}T_r(\rho_A\log\rho_A - \rho_A\log\rho_B)$

$$= T_r(\rho_A'\log\rho_A) - T_r(\rho_A'\log\rho_B) - T_r(\rho_A Z_2')$$

$$= T_r(T_1(\rho_A)\log\rho_A)) - T_r(T_1(\rho_A)\log\rho_B)$$

$$+ \sum_{m=0}^{\infty}T_r(\rho_B^m\rho_A\rho_B^{-m-1}[\log\rho_B, T_2(\rho_B)])$$

Where, $T_k(\rho) = -\iota[H_k, \rho] - \frac{1}{2}\theta_k(\rho)$, $k = 1,2 \ldots$

Special case $\theta_1 = \theta_2 = 0$ (No noise). Then, in this case we find

$$\frac{d}{dt}S(\rho_A, \rho_B) = \iota T_r\{(H_A - H_B)[\rho_A, \log\rho_B]\}$$

When it comes to general terms $\rho_2 = \sum_{\alpha=1}^{p}p_\alpha|e_\alpha\rangle\langle e_\alpha|$ is the spectral representation of ρ_B with $p_\alpha > 0, \quad \forall\alpha$. Then let $X = [\rho_A, \log\rho_B]$, we get

$$\rho_B^N[\rho_A, \log\rho_B]\rho_B^N =$$

$$\sum_{\alpha,\beta=1}^{p}(p_\alpha/p_\beta)^N|e_\alpha\rangle\langle e_\beta|\langle e_\alpha|X|e_\beta|$$

If we assume that $p_\alpha > p_\alpha \Longrightarrow \langle e_\alpha|X|e_\beta\rangle = 0$ then,

$$\lim_{N\to\infty}\rho_B^N[\rho_A, \log\rho_B]\rho_B^{-N}$$

$$= \sum_{\alpha,\beta:p_\alpha=p_\beta}|e_\beta\rangle\langle e_\alpha|X|e_\beta| = \tilde{X}$$

and we then get

$$\frac{d}{dt}S(\rho_A, \rho_B)$$

$$= \iota T_r\{(H_A - H_B)X\} + \iota T_r\{H_B\tilde{X}\}$$

We note that

$$X = [\rho_A(t), \log\rho_B(t)] \equiv X(t)$$

$$= [U_1(t)\rho_A(0)U_1^*(t), U_2(t)\log\rho_B(0)U_2^*(t)]$$

$$= U_1(t)\rho_A(0)U_1^*(t)U_2(t)\log\rho_B(0)U_2^*(t)$$

$$-U_2(t)\log\rho_B(0)U_2^*(t)U_1(t)\rho_A(0)U_1^*(t)$$

Where, $U_1(t) = \exp(-\iota tH_A), U_2(t) = \exp(-\iota tH_B))$

Now assume, $U_1^*(t)U_2(t) \overset{t\to\infty}{\to} \Omega$, (scattering matrix)

Then, $\lim_{t\to\infty} T_r\big((H_A - H_B)X(t)\big)$

$=\lim_{t\to\infty} T_r\{U_2 * (t)U_2(H_A - H_B)U_1(t)\rho_A(0)\Omega\log\rho_B(0)$

Now,

$$\frac{d}{dt}U_2^*(t)U_1(t) = -\iota U_2^*(t)(H_A - H_B)U_1(t)$$

Write, $\Omega = \Omega(\infty) = \lim_{t\to\infty} U_1^*(t)U_2(t)$;

$$\Omega(t) = U_1^*(t)U_2(t).$$

Then, $U_2^*(t)(H_A - H_B)U_1(t) = -\iota\Omega'(t)$,

and we get, $\lim_{t\to\infty} T_r\big((H_A - H_B)X(t)\big)$

$$= \iota T_r\{\Omega'(\infty)\rho_A(0)\Omega(\infty)\log\rho_B(0).$$

If $\Omega'(\infty) = 0$ then this vanishes and we get,

$$\lim_{t\to\infty}\frac{d}{dt}S(\rho_A(t),\rho_B(t)) = \iota T_r\{H_B\tilde{X}(\infty)\}$$

We've seen that

$$\frac{d}{dt}S(\rho_A,\rho_B) = \iota T_r\{H_A[\rho_A,\log\rho_B]$$
$$+\iota\lim_{N\to\infty} T_r\{H_B\rho_B^N[\rho_A,\log\rho_B]\rho_B^{-N}\}$$
$$-\iota T_r\{H_B[\rho_A,\log\rho_B]\}$$

Let, $\Omega(t) = U_1^*(t)U_2(t)$.

Then,

$$[\rho_A,\log\rho_B]$$
$$= U_1(t)\rho_A(0)U_1^*(t)\log\rho_B(0)U_2^*(t)$$
$$= U_1(t)\rho_A(0)\Omega(t)\log\rho_B(0)$$
$$U_2^*(t)$$

So,

$$\iota T_r\{(H_A - H_B)[\rho_A(t),\log\rho_B(t)]\}$$
$$= \iota T_r\{U_2^*(t)(H_A - H_B)U_1(t)\rho_A(0)\Omega(t)\log\rho_B(0)$$
$$= -T_r\{\Omega^{*'}(t)\rho_A(0)\Omega(t)\log\rho_B(0)\}$$

and,

$$T_r\{H_B\rho_B(t)^N[\rho_A(t),\log\rho_B(t)]\rho_B(t)^{-N}\}$$
$$= T_r\{H_BU_2(t)\rho_B(0)^NU_2^*(t)U_1(t)\rho_A(0)$$
$$U_1^*(t)U_2(t)\log\rho_B(0)\rho_B(0)^{-N}U_2^*(t)\}$$
$$-T_r\{H_BU_2(t)\rho_B(0)^N\log(\rho_B(0))U_2^*(t)$$
$$U_1(t)\rho_A(0)U_1(t)U_B(t)\rho_B(0)^{-N}U_2^*(t)$$
$$= T_r\{H_B\rho_B(0)^N\Omega(t)\log\rho_B(0)\rho_B^{-N}$$
$$-T_r\{H_B\rho_B^N\log\rho_B(0)\Omega^*(t)\rho_A(0)\Omega(t)\rho_B^{-N}\}$$

Also note that

$$T_r\{U_2^*(t)(H_A - H_B)$$
$$U_1(t)\rho_A(0)\Omega(t)\log\rho_B(0)\}$$
$$= T_r\{U_2^*(t)U_1(t)H_A$$
$$-H_BU_2^*(t)U_1(t)\rho_A(0)\Omega(t)\log\rho_B(0)\}$$
$$= T_r\{(\Omega^*(t)H_A$$
$$-H_B\Omega^*(t))\rho_A(0)\Omega(t)\log\rho_B(0)\}$$

So,

$$\frac{d}{dt}S\big(\rho_A(t),\rho_B(t)\big) =$$
$$\lim_{N\to\infty}\iota\lim_{N\to\infty} T_r\{H_B\rho_B(0)^N[\Omega^*(t)\rho_A(0)\Omega(t)\log\rho_B(0)]\rho_B^{-N}(0)\}$$
$$+\iota T_r\{(\Omega^*(t)H_A - H_B\Omega^*(t))\rho_A(0)\Omega(t)\log\rho_B(0)\}$$

Let, $\rho_B(0) = \sum_{\alpha=0}^{r} p_\alpha(o)p_\alpha$ be the spectral distinct positions of $\rho_B(0)$. Thus $\{\rho_A(0),..,\rho_r(0)\}$are distinct, $\sum_\alpha p_\alpha(0)T_r(p_\alpha) = 1$, $\sum_1^r p_\alpha = I$, $P_\alpha P_\beta = P_\alpha\delta_{\alpha\beta}$, $P_\alpha^* = P_\alpha$.

In this paper, the rate of change entropy of the two quantum systems is solved and the parameters of the Hamiltonians of the noise operators are determined, which will yield the exact value of the relative entropy of entanglement.

Let us consider,

$$\rho_A'(t) = -\iota[H_A,\rho_A(t)] \text{ and}$$

$$\rho_B'(t) = -\iota[H_A,\rho_B(t)] - \frac{1}{2}\theta(\rho_B(t))$$

Where,

$$\theta(X) = L * LX + XL * L - 2LXL *$$

Then,

$$\frac{d}{dt}T_r(\rho_A\log\rho_A) = 0$$

$$\frac{d}{dt}T_r(\rho_A\log\rho_B)$$
$$= T_r(\rho_A'\log\rho_B) + T_r(\rho_AZ_2')$$

$$Z_2' = \frac{adZ_2}{1 - e^{-adZ_2}}(\rho_B^{-1}\rho_B')$$
$$= adZ_2\sum_{m=0}^{\infty} e^{-m.adZ_2}(\rho_A^{-1}\rho_B')$$
$$= adZ_2\Big(\sum_{m=0}^{\infty}\rho_B^{-m-1}\rho_B'\rho_B^m\Big)$$

So,

$$T_r(\rho_A, Z_2') = \sum_{m\geq 0} T_r(\rho_A\rho_B^{-m-1}[Z_2,\rho_B']\rho_B^m)$$
$$= -\iota\sum_{m\geq 0} T_r\big(\rho_A\rho_B^{-m-1}\big[Z_2,[H_B,\rho_B]\big]\rho_B^m\big)$$
$$-\frac{1}{2}\sum_{m\geq 0} T_r(\rho_A\rho_B^{-m-1}[Z_2,\theta(\rho_B)]\rho_B^m)$$

$$\frac{d}{dt}T_r(\rho_A \log\rho_B) =$$
$$T_r(\rho'_A \log\rho_B) - \iota \sum_{m\geq 0} T_r(\rho_A \rho_B^{-m-1}[Z_2,[H_B,\rho_B]]\rho_B^m)$$
$$-\frac{1}{2}\sum_{m\geq 0} T_r(\rho_A \rho_B^{-m-1}[Z_2,\theta(\rho_B)]\rho_B^m)$$

In conclusion, the change in relative entropy of the two quantum systems is given below;

$$\frac{d}{dt}S(\rho_A,\rho_B) =$$
$$\frac{d}{dt}T_r(\rho_A \log\rho_A - \rho_A \log\rho_B)$$
$$= T_r(\rho'_A \log\rho_A) - T_r(\rho'_A \log\rho_B) - T_r(\rho_A Z'_2)$$
$$\frac{d}{dt}S(\rho_A,\rho_B) = -T_r(\rho'_A \log\rho_B)$$
$$+\iota \sum_{m\geq 0} T_r(\rho_A \rho_B^{-m-1}[Z_2,[H_B,\rho_B]]\rho_B^m)$$
$$+\frac{1}{2}\sum_{m\geq 0} T_r(\rho_B \rho_B^{-m-1}[Z_2,\theta(\rho_B)]\rho_B^m)$$

Where,

$$\theta(\rho_B) = L * L\rho_B + \rho_B L * L - 2L\rho_B L *$$

In order to calculate in time, the initial signal of observable X, the noisy Schrödinger equation is simulated by using a large set of exponential vectors in the Boson-Fock space for noisy baths, and an orthogonal basis for the signal Hilbert space.

IV. Results and Discussions

The observable with the ideal theta value is shown in this section, along with a graphic of the noise to signal energy ratio.

A. The Expectation Value of an Observable of Two Quantum System

This paper is used to determine the average value X, which is expressed by $X = T_r(\rho X)$ of the state of a quantum-mechanical system as described by its density operator ρ with $T_r(\rho)$ =1. The definition of the observable X, which is a 2×2 random Hermitian matrices, as well as the values of H0 and V, constitute the first step in the analysis. It computes this matrix's eigenvalues and accompanying eigenvectors.

By comparing the expected value of an observable in a mixed state with the expected value of the observable in several mutually orthogonal pure states, one can obtain the expected value of the observable in the mixed state [12-15]. Our results are explaining the quantum average measure value to extra the information with effecting of the AWGN and stochastic noise and we will evaluate the performance of our algorithm in the presence of BATH states, that is, compute the noise to signal ratio of the given estimate of $\delta\theta$, that is, $E(\|\delta(\theta) - \delta'(\theta)\|^2)$. Using functional analysis and strict mathematics, it is possible to generate the quantum noise. In particular, we demonstrate how some important stochastic processes from classical probability theory, such as the Brownian motion and Poisson processes, can be viewed as special cases of quantum stochastic processes, which are a family of non-commuting observables in a particular type of Hilbert space called the Boson Fock space when observed in particular states. The randomly generated two Hamiltonian of the given system and find the Eigenvalues through MATLAB.

Algorithm 1: An algorithm of the expectation value of an observable of two quantum system

Data: observable $X \geq 0$
Result: $X = T_r(\rho X)$
Density operator = ρ;
$T_r(\rho) = 1$;
H_0 ← Hermitian Operator;
V ← Hermitian Operator;
Taking the initial state ψ
ψ_0 = rand(3,1) + i * rand(3,1)
For making norm = 1 of ψ
φ ← choose
$\psi_0 = \frac{\psi_0}{\mathrm{norm}(\psi_0)}$
sum ψ = zeros(2, 2, N)
A_1 ← choose
A_2 ← choose
P_1 ← choose
P_1 ← choose
I = 0 ← choose
$Q = (A_1 \oplus P_1) + (P_1 \oplus A_1) + (A_2 \oplus P_2) + (P_2 \oplus A_2) + (A_3 \oplus P_3) + (P_3 \oplus A_3)$
while N ≠ 0 do
    if N is integer value of qubit then
        $X \leftarrow T_r(X^* \times X) \leftarrow \text{minimum};$
        $\theta = (0.5 * T_r(A * A') * \mathrm{real}(f_1 * f'_1) * \mathrm{real}(f_1 * f'_1) * T_r(A^2)))^{-1} * \mathrm{real}(f_1 * T_r(A * Q))$
        is Minimum
    end
    otherwise: $\delta\theta = \frac{\mathrm{real}(T_r(\delta P * Q))}{T_r(Q * Q')} \leftarrow \text{estimating}$
    is Minimum
end

Where,

$$\delta P = \delta\theta * Q$$

And,

H_A = rand (2,2) + j*rand (2,2);

H_A= [0.8178.   0.4275 - 0.0756i;

0.4275 + 0.0756i   0.0225]

The Hermitian matrix equation of the Hamiltonian of the first system is given below;

$H_A=(H_A+H_A')/2$

H_B=rand (2,2) + j*rand (2,2);

H_B=[0.4229   0.3464 - 0.0307i;

0.3464 + 0.0307i   0.4709]

$\delta X =$

-0.070838082586555 - 0.000000000000000i	-0.015815573941431 + 0.007750834345286i	-0.015815573941431 + 0.007750834345286i	0.071195946682088 - 0.010664577493442i
-0.015815573941431 - 0.007750834345286i	0.070838082586553 + 0.000000000000000i	0.070838082586553 + 0.000000000000000i	0.015815573941431 - 0.007750834345286i
-0.015815573941431 - 0.007750834345286i	0.070838082586553 + 0.000000000000000i	0.070838082586553 + 0.000000000000000i	0.015815573941431 - 0.007750834345286i
0.071195946682088 + 0.010664577493442i	0.015815573941431 + 0.007750834345286i	0.015815573941431 + 0.007750834345286i	-0.070838082586553 + 0.000000000000000i

The Hermitian Matrix equation of the Hamiltonian of the second system is given below;

$$H_B=(H_B+H_B')/2;$$

First, initialize the states and choose the value of A_1, A_2, P_1 and P_2 for estimating the theta. So, the estimated value of θ is 0.33 for 2×2 matrices, additionally, if we increase the size of the qubit, the estimated values of θ is 0.28 for 3×3. Then second, we are designed δX for 2×2 matrices, consider δθ=0.33 with a random noise generating from random AWGN.

Since the collapse postulate is taken into account, continuous measurement is not covered in this section. A single measurement is known to cause the system's state to collapse to the eigenstate of the measured observable, which corresponds to the observed result. Since the metric above only displayed the minimal value of the observable, it is clear that, for each time index of T, the value of our error energy function approaches zero or its smallest value.

B. NSER (Noise- to-Signal Energy Ratio) of Entropy

We can use the NSER as a performance index to calculate the impact of this noise on the error energy. This chart, which is displayed below, shows how noise to signal energy ratios can remain low over an extended length of time, which verifies a successful performance. Using the Frobenius norm, we must minimize the error function or cost function.

The error function has to be determined the optimum operators L_1, L_2, S, so that $\int_0^T \parallel \rho_s(t,u) - \rho_d(t) \parallel^2 dt$ is a minimum. Assuming the function $u(t)$,$0 \le t \le T$ and H given.

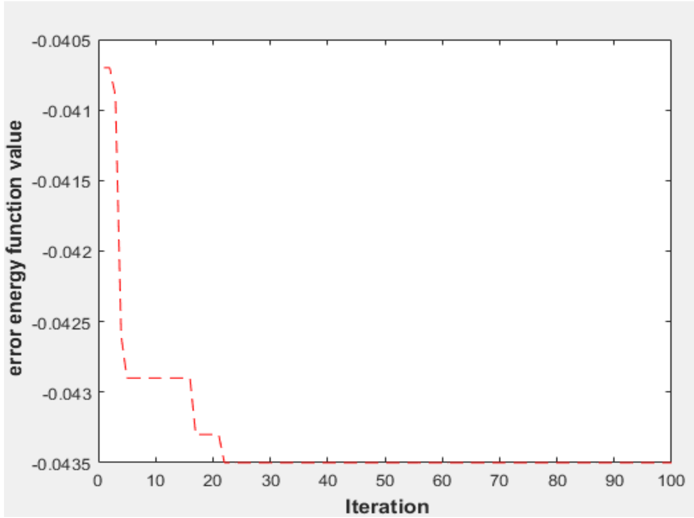
$$\text{NSER} = \frac{\|\rho_s(t,u)-\rho_d(t)\|^2}{\rho_d(t)2} = \frac{\zeta_{min}}{\rho_d(t)2}$$

Algorithm 2: NSER (Noise – to – Signal Energy Ratio) of entropy

Data: Using the Frobenius norm, we must minimize the error function or cost function.
Result: NSER should be remains less than unity.
initialization;
while *While* condition **do**
  instructions;
  if *condition* **then**
    instructions1: $\delta\theta$ is minimum;
    instructions1: θ is minimum;
    then
    $X \leftarrow$ minimum
    then
    instructions3: $\rho \leftarrow$ minimum
  then $NSER = s(t,u) - d(t)^2)/(d(t)) = min/(d(t))$
  else
    instructions4: NSER is minimum;
  Outcome: Rate of the change of Entropy is minimum
    Final:
    $\frac{dS(\rho_A-\rho_B)}{dt} = \frac{dT_{r(\rho_A \log(\rho_A)-\rho_A \log(\rho_B))}}{dt}$
  end
end

Where, ζ_{min} is the error energy function value and $\rho_d(t)$ is the desired density function. We show through simulation that the NSER of entropy stabilizes to a small value, supporting the information inequality that states conditionally reducing entropy decreases information. A better design, one with a lower SER, might theoretically be obtained by first averaging over the noise distribution and then minimizing with respect to the nonrandom functions (see Fig. 1).

Noise to signal Energy ratio



Time

Fig. 1. Noise-to-signal energy ratio plot for quantum system

Since reducing noise effects and highlighting the signal process is the whole purpose of collecting measurements, the NSER should gradually decrease with time.

$$\frac{d}{dt}S(\rho_A,\rho_B)=\frac{d}{dt}T_r(\rho_A\log\rho_A-\rho_A\log\rho_B)$$

We compute NSER, and it remains less than unity. We also showed how to use stochastic differential equations to calculate the relative entropy of two quantum systems plus bath density, i.e. stochastic system density. Through simulations, we are justifying that the rate of change of relative entropy stabilizes to a value less then $\theta(\rho_B)$, which realistically justifies the information inequality stating that conditioning reduces entropy. The idea will be helpful to research communities in applied mathematics, physics, and quantum information theory who seek to investigate the variety of applications of classical and quantum stochastics to issues of physics and engineering, to sum up the conclusion.

V. Conclusions

We have determined the quantum relative entropy rate between two mixed states using noisy Schrödinger equations with varied Hamiltonians and Lindbland operators. For our calculations, we applied the conventional formula for calculating the exponential map of matrices. We may compute the rate of relative entropy as the asymptotic limit $t \to \infty$ based on the scattering matrices connected to the pair of Hamiltonians that generate the two states. It is important to look into the circumstances in which the asymptotic relative entropy rate for the Hamiltonian and Lindbland operators continues to be below the specified threshold. The asymptotic limit in this scenario would ensure that there is a short gap between the two states. Conditioning is known in classical information theory to decrease entropy, specifically $H(X/Y) \leq H(X)$. As a result, we anticipate that in the quantum setting, the entropy of the filtered state will be reduced using HP equations based on detecting the noise process. In further work, we will also extend this formalism to Belavkin's quantum filtering theory based on the Hudson-Parthasarathy quantum stochastic by demonstrating that when this equation for a particle travelling in a potential with damping and noise is characterized in terms of the Wigner distribution function, then it is exactly the same as the Kushner-Kallianpur stochastic filter but with quantum correction terms stated as a power series in Planck's constant.

References

[1] Shapiro J.H., Zhang Z., Wong F.N.C.: Secure communication via quantum illumination. Quantum Information Processing, 13(10), (2014)

[2] Dirac P.A.M.: The principles of quantum mechanics, Oxford university press, 4th edition, New York,(1958)

[3] Schrödinger E.: An Undulatory Theory of the Mechanics of Atoms and Molecules. Phys. Rev., 28,(1926)

[4] Griffiths D.J.: Introduction to quantum mechanics, Pearson Hall, 2nd edition, USA, (2005).

[5] Christopher A. F., Carlton M. C.: Mathematical techniques for quantum communication theory. Open Systems and Information Dynamics, 3(3), 345-356(1995)

[6] Garg N., Parthasarathy H., Upadhyay D.K.: "Real-time simulation of H–P noisy Schrödinger equation and Belavkin filter", Quantum Information Processing, 10.1007/s11128-017-1572-4, (2017).

[7] Von Neumann J.: Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press, Princeton, NJ, (1955).

[8] Helstrom C.W.: Detection Theory and Quantum Mechanics. Information and Control, 10, 254-291(1967)

[9] Helstrom C.W.: Detection Theory and Quantum Mechanics (II). Information and Control, 13, 156-171(1968)

[10] Peyton B., Sard E., Lange R., Arams F.R.: On Optimal Quantum Receivers for Digital Signal Detection. Proceedings of the IEEE, (1970)

[11] Li K., Zuo Y., Zhu B.: Suppressing the Errors Due to Mode Mismatch for M-ary PSK Quantum Receivers Using Photon-Number-Resolving Detector. IEEE Photon. Technol. Lett., 25(22),(2013)

[12] Shi J., Shi R., Guo Y., Peng X., Lee M. H.: Probabilistic quantum relay communication in the noisy channel with analogous space-time code. Quantum Information Processing., 12, 1859-1870(2013)

[13] Kato K., Hirota O.: Square-Root Measurement for Quantum Symmetric Mixed State Signals. IEEE Transaction on Information Theory, 49(12), (2003)

[14] Atmanspacher H., Kurths J., Scheingraber H., Wackerbauer R., Witt A.: Complexity and meaning in nonlinear dynamical systems, Open Systems and Information Dynamics, 1(2), 269-289(1992)

[15] Holevo A.S.: Statistical Decision Theory for Quantum Systems. Journal Of Multivariate Analysis, 3, 337-394(1973)

[16] Yuen H.P., Kennedy R.S., Lax M.: Optimum Testing of Multiple Hypotheses in Quantum Detection Theory. IEEE Transaction on Information Theory, 21(2), (1975)

[17] Helstrom C.W.: Bayes-Cost Reduction Algorithm in Quantum Hypothesis Testing. IEEE Transaction on Information Theory 28(2), 359-366 (1982)

[18] Kennedy R.S.: A near-optimum receiver for the binary coherent state quantum channel. Research Lab. Electron MIT Cambridge Tech. Rep., 108, (1973)

[19] Dolinar S.J.: An optimum receiver for the binary coherent state quantum channel. Research Lab. Electron MIT Cambridge Tech. Rep., 111, (1973)

[20] Sasaki M., Hirota O.: Optimum decision scheme with a unitary control process for binary quantum-state signals. Phys. Rev. A, 54, (1996)

[21] Wang D., Huang A.J., Sun W.Y., Shi J.D., Ye L.: "Practicle single-photon-assisted remote state preparation with non-maximally entenglement", Quantum Information Processing, (15), 3367-3381, (2016).

[22] Wang, D., Hu, Y. D., Wang, Z. Q., Ye, L.:" Efficient and faithful remote preparation of arbitrary three-and four-particle W-class entangled states". Quantum Information Processing, 14(6), 2135-2151, (2015).

[23] Lo H.K.: "Classical-communication cost in distributed quantum-information processing: a generalization of quantum-communication complexity" Physical Review A, (62), (2000).

[24] Wang D., Hoen R.D., Ye L., Kais S.: "Generalized Remote Preparation of Arbitrary m-qubit Entagled States via Genuine Entanglements", Entropy (17), 1755-1774, (2015)

[25] Hayashi M., Universal Coding for Classical-Quantum Channel, Communication in Mathematical Physics 289, 1087-1098, (2009)

[26] Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information, pp. 171–286. Cambridge University Press, Cambridge (2001).

[27] Parthasarathy K.R.: "Coding theorems of classical and quantum Information theory", Hindustan Book Agency; 2013.

[28] Dowker F., Tabatabai Y.G.: Dynamical wavefunction collapse models in quantum measure theory. J. Phy. A Mathematical and Theoritical, 41(20), (2008)

[29] Mensky M., Audretsch J.: Continuous QND measurements: no quantum noise. Journal Applied Physics B, 64(2), 129-136, (1997)

[30] Srikanth R.: A Computational Model for Quantum Measurement. Quantum Information Processing, 2(3), 153-199, (2003)

[31] Gisin N., Popescu S., Scarani V., Wolf S. and Wullschleger J.: Oblivious transfer and quantum channels as communication resources. Natural Computing, 12(1), 13-17, (2013).

[32] Karlsson A., Björk G.: Quantum correlations in dual quantum measurements. Journal Applied Physics B, 64(2), 235-241, (1997)

[33] Prajapati, Bhaveshkumar B. and Chaubey, N. K., “Realization of Relative Entropy Evolution in the Sudarshan-Lindblad for two Quantum Systems”, JOURNAL OF ALGEBRAIC STATISTICS, Volume 13, No. 1, 2022, p. 490-497.

[34] Prajapati, Bhaveshkumar B. and Chaubey, N. K., “Quantum Key Distribution: The Evolution” Quantum Crypstography and the Future of Cyber Security (IGL Global), 2020 |P.: 15.