A quadratic convergence method for the management equilibrium model

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***Abstract*—in this paper, we study a class of methods for solving the management equilibrium model. We first give an estimate of the error bound for the model, and then, based on the estimate of the error bound, propose a method for solving the model. We prove that our algorithm is quadratically convergent without the requirement of existence of a non-degenerate solution.**

Keywords—Management equilibrium model; estimation of error bound; algorithm; quadratic convergence

I. INTRODUCTION

The management equilibrium model seeks a vector $(x^*, y^*) \in R^{2n}$ such that

$$x^* \geq 0, y^* \geq 0, (x^*)^{\mathrm{T}} y^* = 0, Mx^* - Ny^* = Qz^* + q, \quad (1)$$

Where $M, N \in R^{m\times n}$, $Q \in R^{m\times l}$, $q \in R^m$, and there exists $z^* \in R^l$. The model originated from equilibrium problems in economic management, engineering, etc. Applications of complementarity problems from the field of economics include general Walrasian equilibrium, spatial price equilibria, invariant capital stock, market equilibrium, optimal stopping, and game-theoretic models, In engineering, the complementarity problems also plays a significant role in contact mechanics problems, structural mechanics problems, obstacle problems mathematical physics, Elastohydrodynamic lubrication problems, traffic equilibrium problems(such as a pathbased formulation problem, a multicommodity formulation problem, network design problems),etc. [1,2] For example, the equilibrium of supply and demand in an economic system is often depicted as a complementary model between two decision variables. As another example, the typical Walras' Law of competition equilibrium in economic transactions can also be converted to complementary model between price and excess demand [3].

Recently, many effective methods have been proposed to solve (1) [4−6]. The basic idea of these methods is to convert (1) into an unconstrained or a simply constrained optimization problem. As we known, if the Jacobian matrix at a solution to (1) is non-singular, then it is guaranteed that the Levenberg-Marquardt (L-M) algorithm is quadratically convergent [5,6]. Lately, Yamashita and Fukushima have proved that the condition for the local error bound to hold is weaker than the non-singularity of the Jacobian matrix [7]. This motivates the establishment of an error bound for (1). The establishment of LCP error bound has been extensively studied (see literature review [8]). For example, Mangasarian and Ren have given an error bound under the R_0 - matrix condition [9]. Clearly, (1) is a generalization of LCP, which prompts whether or not the LCP error bound can be generalized to (1). For this reason, we focus on the establishment of an error bound for (1), design a smooth algorithm for solving (1) using the error bound, and analyze the convergence of the algorithm as well as the rate of convergence.

In section 2, we give primarily an equivalent conversion of (1). In section 3, using a new residual function, we establish an error bound for (1) under more general conditions. In section 4, based on the established error bound, we propose a smooth algorithm for solving (1), and prove that the given algorithm is quadratically convergent without the requirement of existence of a non-degenerate solution. Compared with the convergence of algorithms in [5,6], the condition is weaker.

Now we give some notations. The inner product of vectors $x, y \in R^n$ is written as $x^{\mathrm{T}} y$. Let $\|\cdot\|$ be the Euclidean norm. For ease of presentation, we write (x, y, z) for column vector $(x^{\mathrm{T}}, y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$, and use $dist(\xi, \omega^*)$ for the shortest distance from vector ξ to a closed convex set ω^*.

II. EQUIVALENT CONVERSION OF THE MANAGEMENT EQUILIBRIUM MODEL

We give in this section an equivalent conversion of (1). For convenience, let $\xi^* \square (x^*, y^*, z^*) \in R^{n+n+l}$. Then, (1) can be converted equivalently to the following problem : Find ξ^* such that

$$\begin{cases} A\xi^* \geq 0, B\xi^* \geq 0, \\ (A\xi^*)^T (B\xi^*) = 0, \\ (M, -N, -Q)\xi^* - q = 0, \end{cases} \quad (2)$$

Where $A = (I, 0, 0), B = (0, I, 0)$. Let ω^* be the set of solutions of（2） and assume that it is nonempty.

We have the following conclusion.

Theorem 1.1 Vector $(x^*, y^*) \in R^{2n}$ is a solution to (1) if and only if there exists $z^* \in R^l$ such that $\xi^* = (x^*, y^*, z^*)$ is a solution to (2).

III. Estimation Of The Error Bound Of The Management Equilibrium Model

This section mainly establishes the error bound for the management equilibrium model. First, we give some related results, the definition of projection operator and it related properties.

Theorem 2.1 For a given positive constant ρ, there exists a constant $\eta_1 > 0$ such that

$$dist(\xi, \omega^*) \le \eta_1 r(\xi), \forall \xi \in \omega, \|\xi\| \le \rho,$$

where $r(\xi) = \| \min\{A\xi, B\xi\|$,

$\omega = \{\xi \in R^{2n+l} \mid (M, -N, -Q)\xi = q\}$.

Proof. Assume that the theorem does not hold. Then there exists a sequence $\{\xi^k\}$, such that for any positive integer k, we have

$$dist(\xi^k, \omega^*) > kr(\xi^k) \ge 0.$$

That is,

$$\frac{r(\xi^k)}{dist(\xi^k, \omega^*)} \to 0, k \to \infty, \qquad (3)$$

where $\xi^k \in \omega$, and $\|\xi^k\| \le \rho$. Since sequence $\{\xi^k\}$ is bounded, and $r(\xi)$ is continuous, together with (3), we have $r(\xi^k) \to 0, k \to \infty$. In addition, sequence $\{\xi^k\}$ has a convergent subsequence $\{\xi^{k_i}\}$. Let $\xi^{k_i} \to \bar{\xi}(k_i \to \infty)$, where $\bar{\xi} \in \omega^*$. We have the following conclusion.

$$\frac{r(\xi^{k_i})}{\| \xi^{k_i} - \bar{\xi} \|} \to \beta(k_i \to \infty). \qquad (4)$$

Where, β is a positive constant.

On the other hand, from (3) we have

$$\frac{r(\xi^{k_i})}{\| \xi^{k_i} - \bar{\xi} \|} \le \frac{r(\xi^{k_i})}{dist(\xi^{k_i}, \omega^*)} \to 0(k_i \to \infty).$$

This contradicts with (4), hence the theorem is proved. □

We give in the following the error bound established by Hoffman [10].

Lemma 2.1 For a polyhedral cone

$$P = \{x \in R^n \mid D_1 x = d_1, D_2 x \le d_2\}$$

, where $D_1 \in R^{l\times n}, D_2 \in R^{m\times n}$, $d_1 \in R^l$, $d_2 \in R^m$, there exists a constant $c > 0$, such that

$$dist(x, P) \le c[\| D_1 x - d_1 \| + \| (D_2 x - d_2)_+ \|], \forall x \in R^n.$$

Now, we also give the definition of projection operator and its related properties [11]. For a nonempty closed convex set $S \subseteq R^n$, the orthogonal projection from vector $x \in R^n$ onto S is

$$P_S(x) := \arg\min\{\| y - x \| \mid y \in S\},$$

and it has the following property.

Lemma 2.2 For any vectors $u, v \in R^n$, we have

$$\| P_s(u) - P_s(v) \| \le \| u - v \|.$$

Using Theorem 2.1, Lemma 2.1 and Lemma 2.2, we have the main result.

Theorem 2.2 For any positive constant ρ, there exists a constant $\eta_2 > 0$ such that

$$dist(\xi, \omega^*) \le \eta_2(\| (M, -N, -Q)\xi - q \| + r(\xi)), \quad \|\xi\| \le \rho,$$

Where $r(\xi) = \| \min\{A\xi, B\xi\|$.

Proof. For any vector $\xi \in R^{2n+l}$, there exists $\bar{\xi} \in \omega$, such that $\| \xi - \bar{\xi} \| = dist(\xi, \omega)$. From Lemma 2.1, there exists a constant $c_1 > 0$, such that

$$dist(\xi, \omega) \le c_1 \| (M, -N, -Q)\xi - q \|.$$

Furthermore,

$$\begin{aligned}
&\| r(\xi) - r(\bar{\xi}) \| \\
&= \| \min\{A\xi, B\xi\} - \min\{A\bar{\xi}, B\bar{\xi}\} \| \\
&= \| (A\xi - P_{R_+}(A\xi - B\xi)) - [A\bar{\xi} - P_{R_+}(A\bar{\xi} - B\bar{\xi})] \| \\
&\le \| A(\xi - \bar{\xi}) \| + \| (P_{R_+}(A\xi - B\xi) - P_{R_+}(A\bar{\xi} - B\bar{\xi})) \| \\
&\le \| A(\xi - \bar{\xi}) \| + \| (A\xi - B\xi) - (A\bar{\xi} - B\bar{\xi}) \| \\
&\le 2 \| A(\xi - \bar{\xi}) \| + \| B\xi - B\bar{\xi} \| \\
&\le (2 \| A \| + \| B \|) dist(\xi, \omega),
\end{aligned}$$

Where the second inequality is based on Lemma 2.2. Combined with the above formula, we have

$$\| r(\bar{\xi}) \| \le \| r(\xi) \| + (2 \| A \| + \| B \|) dist(\xi, \omega) \qquad (5)$$

From (5) and Theorem 2.1, we have

$$\begin{aligned}
&dist(\xi,\omega)\\
&\le dist(\xi,\omega)+dist(\bar{\xi},\omega^*)\le dist(\xi,\omega)+\eta_1 r(\bar{\xi})\\
&\le dist(\xi,\omega)+\eta_1[r(\xi)+(2\|A\|+\|B\|)dist(\xi,\omega)]\\
&\le[\eta_1(2\|A\|+\|B\|)+1]dist(\xi,\omega)+\eta_1 r(\xi)\\
&\le[\eta_1(2\|A\|+\|B\|)+1]c_1\|(M,-N,-Q)\xi-q\|)\ +\eta_1 r(\xi)\\
&\le\eta_2[\|(M,-N,-Q)\xi-q\|)+r(\xi)],
\end{aligned}$$

Where $\eta_2=\max\{[\eta_1(2\|A\|+\|B\|)+1]c_1,\eta_1\}.$ □

In the following we using Fischer function ([12]) to establish another error bound. Define $\phi: R^2\to R^1$ and

$$\phi(a,b)=\sqrt{a^2+b^2}-a-b,\forall a,b\in R.$$

It has the following property:

$$\phi(a,b)=0\Leftrightarrow a\ge 0,b\ge 0,ab=0,$$

In addition, Tseng[13] gives the following conclusion.

Lemma 2.3

$$[2-\sqrt{2})]|\min(a,b)|\le|\phi(a,b)|$$
$$\le(\sqrt{2}+2)|\min(a,b)|.$$

For any vectors $a,b\in R^n$, define a vector-valued function $\Psi(a,b)=(\phi(a_1,b_1),\phi(a_2,b_2),...,\phi(a_n,b_n))$. Based on this mapping, (2) can be converted into the following equation

$$\Phi(\xi):=\begin{pmatrix}\Psi(A\xi,B\xi)\\(M,-N,-Q)\xi-q\end{pmatrix}=0,$$

Clearly, using Lemma 2.3 and Theorem 2.2, it is easy to have the following result.

Theorem 2.3 For any given positive constant ρ, there exists a constant $\eta_3>0$ such that

$$dist(\xi,\omega^*)\le\eta_3\|\Phi(\xi)\|,|\xi|\le\rho.$$

As function $\Phi(x)$ is not smooth, let $\Psi_t: R^2\to R$ denote smooth Fisher-Burmeister function

$$\Psi_t(a,b)=\sqrt{a^2+b^2+2t^2}-a-b,$$

Where $t>0$ is a smooth parameter. For ease of presentation, let

$$\Gamma(x,y,t)=(\psi_t(x_1,y_1),...,\Psi_t(x_n,y_n))^T,$$

Where $x=(x_1,...,x_n)^T, y=(y_1,...,y_n)^T$,

And $p(a,b,t)=\Psi_t(a,b)$. We define mapping

$$F: R^{2n+l}\times(0,+\infty)\to R^{n+m}\times(0,+\infty),$$

That is,

$$F(\xi,t)=\begin{pmatrix}\Gamma(A\xi,B\xi,t)\\(M,-N,-Q)\xi-q\\t\end{pmatrix}.$$

Let $f(\xi,t)=F(\xi,t)^T F(\xi,t)=\|F(\xi,t)\|^2$.

Obviously, $\xi^*\in\omega^*\Leftrightarrow(\xi^*,0)$ is a solution to $F(\xi,t)=0$.

Therefore we construct a smooth method to solve $F(\xi,t)=0$, and assume that the set of solutions to $F(\xi,t)=0$ is ω_t^*.

First we give the following properties of $p(a,b,t)$ [14,15].

Lemma 2.4 Function $p(a,b,t)$ has the following properties:

a) On $R^2\times(0,+\infty)$, function $p(a,b,t)$ is continuously differentiable, and strongly semi-smooth, that is,

$$p(a+\Delta a,b+\Delta b,t+\Delta t)-p(a,b,t)$$
$$-V^T(\Delta a,\Delta b,\Delta t)=O\|(\Delta a,\Delta b,\Delta t)\|^2,$$
$$\forall(a,b,t)\in R^2\times[0,+\infty),$$

Where $V\in\partial p(a+\Delta a,b+\Delta b,t+\Delta t)$, and ∂p is the Clarke generalized gradient of p.

b) $\forall(a,b,t)\in R^2\times(0,+\infty)$, we have

$$|\Psi(a,b)-p(a,b,t)|$$
$$=|\Psi(a,b)-\Psi_t(a,b)|\le\sqrt{2}\ t.$$

Based on Lemma 2.4, we have the following result.

Theorem 2.4 Function $F(\xi,t)$ has the following properties:

a) On $R^{2n+l}\times(0,+\infty)$, function $F(\xi,t)$ is continuously differentiable, locally Lipschitz continuous, and strongly semi-smooth, that is, there exist constants $L_1>0$, $L_2>0$, $b_1>0$ such that

$$\|F(\xi+\Delta\xi,t+\Delta t)-F(\xi,t)\|\le L_1\|(\Delta\xi,\Delta t)\|, \tag{6}$$

$$\begin{aligned}&\|F(\xi+\Delta\xi,t+\Delta t)-F(\xi,t)-H^T(\Delta\xi,\Delta t)\|\\&\le L_2\|(\Delta\xi,\Delta t)\|^2,\forall(\xi,t)\in R^{2n+l}\times(0,+\infty),\\&H\in\partial F(\xi+\Delta\xi,t+\Delta t),\end{aligned} \tag{7}$$

$$\forall(\Delta\xi,\Delta t)\in N(0,b_1)$$
$$=\{(\Delta\xi,\Delta t)\,|\,\|(\Delta\xi,\Delta t)\|\le b_1, t+\Delta t\ge 0\},$$

Where $\partial F(\xi,t)$ is the Clarke generalized gradient of $F(\xi,t)$.

b) For $(\xi^*,0)\in\omega_t^*$*, there exists a neighbourhood*

$$N((\xi^*0),b_2)=\{(\xi,t)\,|\,\|(\xi,t)-(\xi^*0)\|\le b_2, t\ge 0\},$$

And a constant $c_1>0$, for any

$$(\xi,t)\in N((\xi^*,0),b_2),$$

We have

$$dist((\xi,t),\omega_t^*)\le c_1\|F(\xi,t)\|. \quad (8)$$

Proof. The result of (i) follows from Lemma 2.1 directly.

(ii) For any $|\xi|\le\rho$, there exists a constant $b_3>0,$ such that

$$dist(\xi,\omega^*))\le\eta_2\|\Phi(\xi)\|,$$
$$\forall\xi\in N(\xi,b_2)=\{\xi\,|\,\|\xi-\xi^*\|\le b_2\},$$

Let $dist(\xi,\omega^*))=\|\xi-\bar{\xi}\|,$ where $\bar{\xi}\in\omega^*$.

From Lemma 2.4(ii), we have

$$|\,\|\Phi(\xi)\|-\|\Phi_t(\xi)\|\,|\le\|\Phi(\xi)-\Phi_t(\xi)\|$$
$$\le\sqrt{2n}\ t,$$

Where $\Phi_t(\xi):=\begin{pmatrix}\Psi_t(A\xi,B\xi)\\(M,-N,-Q)\xi-q\end{pmatrix},$ for any

$$(\xi,t)\in N((\xi^*,0),b_2)=\{(\xi,t)\,|\,\|(\xi,t)-(\xi^*,0)\|\le b_2\},$$

We have

$$dist((\xi,t),\omega^*))\le\left\|(\xi,t)-(\bar{\xi},0)\right\|\le\left\|\xi-\bar{\xi}\right\|+t$$
$$\le\eta_2\|\Phi(\xi)\|+t\le\eta_2\|\Phi_t(\xi)\|+(\sqrt{2n}\eta_2+1)t$$
$$\le(\sqrt{2n}\eta_2+1)(\|\Phi_t(\xi)\|+t)$$
$$\le\sqrt{2}(\sqrt{2n}\eta_2+1)(\sqrt{\|\Phi_t(\xi)\|^2+t^2})$$
$$\le\sqrt{2}(\sqrt{2n}\eta_2+1)\left\|F(\xi,t)\right\|,$$

where $c_1=\sqrt{2}(\sqrt{2n}\eta_2+1).$ □

IV. Algorithm And Convergence

In this section, we give a smooth and convergent algorithm for solving (1), and using the error bound established in section 2, prove the quadratic convergence of the given smooth algorithm without the condition of existence of a non-degenerate solution.

Algorithm 3.1

Step 1: Choose parameters $\sigma\in(0,1),\ \rho>0$ and $\varepsilon\ge 0$, initial value $(\xi,t)^0\in R^{2n+l+1}$, and $|(\xi,t)^0|\le\rho$. Let $k=0$.

Step 2: Stop if $\|\nabla f(\xi^k,t^k)\|\le\varepsilon$; otherwise, turn to Step 3.

Step 3: Choose the Jacobian matrix H^k of $F(\xi^k,t^k)$, and let $d^k=(\Delta\xi^k,\Delta t^k)$ be the solution to the following strict quadratic programming

$$\min\quad \theta^k(d)$$
$$s.t.\quad \left\|(\xi,t)^k+d\right\|\le\rho,\ |\Delta t|\le\frac{1}{1+\mu^k}t^k \quad (9)$$

Where

$$\theta^k(d)=\|F(\xi^k,t^k)+H^kd\|^2+\mu^k\|d\|^2,$$
$$\mu^k=\sigma\|F(\xi^k,t^k)\|^2.$$

Step 4: Let $\xi^{k+1}:=\xi^k+\Delta\xi^k,$

$t^{k+1}:=t^k+\Delta t^k,\ k:=k+1$, turn to Step 2.

In the following convergence analysis, assume that Algorithm 3.1 generates an infinite sequence. We have the following result.

Theorem 3.1 Assume that Algorithm 3.1 generates a sequence $\{(\xi^k,t^k)\}$. If the initial value is close sufficiently to $\{(\xi^*,0)\}$, which is a solution to $F(\xi,t)=0$, then $\{\mathrm{dist}((\xi^k,t^k),\omega^*)\}$ converges quadratically to 0, i.e., sequence $\{\xi^k\}$ converges quadratically to $\bar{\xi}\in\omega^*$.

Proof. Let $\tau:=(\xi,t)$, $\tau^*:=(\xi^*,0)$. For any tiny $\delta>0$, define

$$B_\delta(\tau^*):=\begin{Bmatrix}(\xi,t)\in R^{2n+l}\times(0,+\infty),\\ \left\|(\xi,t)-(\xi^*,0)\right\|\le\delta\end{Bmatrix},$$

In the following we prove the theorem in three steps.

First we prove the following result.

If $\tau^k\in B_{\delta/2}(\tau^*)$, then

$$\left\|d^k\right\| \le c_2 dist(\tau^k, \omega_t^*), \quad (10)$$

$$\left\|F(\tau^k) + H^k d^k\right\| \le c_3 dist(\tau^k, \omega_t^*)^2, \quad (11)$$

Where $c_2 > 0, c_3 > 0$ are constants.

Let the closest point in ω_t^* to τ^k be $\bar{\tau}^k$, that is,

$$\| \tau^k - \bar{\tau}^k \| = dist(\tau^k, \omega_t^*) \quad (12)$$

Let $\bar{d}^k = \tau^k - \bar{\tau}^k$. As d^k is the globally optimal solution to (9), we have

$$\theta^k(d^k) \le \theta^k(\bar{d}^k) = \theta^k(\tau^k - \bar{\tau}^k). \quad (13)$$

Since $\tau^k \in B_{\delta/2}(\tau^*)$, we have

$$\| \bar{\tau}^k - \tau^* \| \le \| \bar{\tau}^k - \tau^k \| + \| \tau^k - \tau^* \|$$
$$\le \| \tau^k - \tau^* \| + \| \tau^k - \tau^* \| \le \delta.$$

Hence, $\bar{\tau}^k \in B_\delta(\tau^*)$. From the definition of μ^k, (8) and (12), we have

$$\mu^k = \sigma \| F(\tau^k) \|^2 \ge \sigma c_1^{-2} dist(\tau^k, \omega_t^*)^2$$
$$= \sigma c_1^{-2} \| \tau^k - \bar{\tau}^k \|^2 \quad (14)$$

Using (12) - (14) and (7), together with the definition of $\theta^k(d)$, we know that

$$\| d^k \|^2 \le [1/u^k]\theta(d^k) \le [1/u^k]\theta(\tau^k - \bar{\tau}^k)$$
$$= [1/u^k][\| F(\tau^k) + H^k(\tau^k - \bar{\tau}^k) \|^2 + \mu^k \| \tau^k - \bar{\tau}^k \|^2]$$
$$= [1/u^k] \| F(\tau^k) + F(\bar{\tau}^k) + H^k(\tau^k - \bar{\tau}^k) \|^2$$
$$+ \| \tau^k - \bar{\tau}^k \|^2 \le [1/u^k] L_2^2 \| \tau^k - \bar{\tau}^k \|^4 + \| \tau^k - \bar{\tau}^k \|^2$$
$$\le \left\{1/\left[\sigma c_1^{-2} \| \tau^k - \bar{\tau}^k \|^2\right]\right\} L_2^2 \| \tau^k - \bar{\tau}^k \|^4 + \| \tau^k - \bar{\tau}^k \|^2$$
$$\le \left\{L_2^2/\left[\sigma c_1^{-2}\right]\right\} \| \tau^k - \bar{\tau}^k \|^2 + \| \tau^k - \bar{\tau}^k \|^2$$
$$\le \left\{L_2^2/\left[\sigma c_1^{-2}\right] + 1\right\} \| \tau^k - \bar{\tau}^k \|^2 \le c_2 dist(\tau^k, \omega_t^*)^2$$

Where $c_2 = \left\{L_2^2/\left[\sigma c_1^{-2}\right] + 1\right\}$. Then (10) holds.

From the definition of $\theta^k(d)$, we know

$$\| F(\tau^k) + H^k d^k \|^2 \le \theta^k(d^k). \quad (15)$$

In addition, from (13), (7) and the definition of $\theta^k(d)$, we have

$$\theta^k(d^k) \le \theta^k(\tau^k - \bar{\tau}^k) \le \| F(\tau^k) + F(\bar{\tau}^k) + H^k(\tau^k - \bar{\tau}^k) \|^2$$
$$+ \mu^k \| \tau^k - \bar{\tau}^k \|^2 \le L_2^2 \| \tau^k - \bar{\tau}^k \|^4 + \mu^k \| \tau^k - \bar{\tau}^k \|^2 \quad (16)$$

From (6), we also have

$$\mu^k = \sigma \| F(\tau^k) \|^2 = \sigma \| F(\tau^k) - F(\bar{\tau}^k) \|^2$$
$$\le \sigma L_1^2 \| \tau^k - \bar{\tau}^k \|^2$$

Together with (15) and (16), we have

$$\| F(\tau^k) + H^k d^k \|^2 \le \theta^k(d^k)$$
$$\le L_2^2 \| \tau^k - \bar{\tau}^k \|^4 + \mu^k \| \tau^k - \bar{\tau}^k \|^2$$
$$\le L_2^2 \| \tau^k - \bar{\tau}^k \|^4 + \sigma L_1^2 \| \tau^k - \bar{\tau}^k \|^4$$
$$\le (L_2^2 + \sigma L_1^2) \| \tau^k - \bar{\tau}^k \|^4$$
$$= c_3^2 \| \tau^k - \bar{\tau}^k \|^4 = c_3^2 dist(\tau^k, \omega_t^*)^4$$

Where $c_3 = \sqrt{L_2^2 + \sigma L_1^2}$.

Next, for any natural number k, if τ^k, $\tau^{k-1} \in B_{\delta/2}(\tau^*)$, there exists $c_4 > 0$, such that

$$dist(\tau^k, \omega_t^*) \le c_4 dist(\tau^{k-1}, \omega_t^*)^2 \quad (17)$$

In fact, since τ^k, $\tau^{k-1} \in B_{\delta/2}(\tau^*)$, and $\tau^k = \tau^{k-1} + d^{k-1}$, together with (8), we have

$$\| F(\tau^{k-1} + d^{k-1}) \| - \| F(\tau^{k-1}) + H^{k-1} d^{k-1} \|$$
$$\le \| -F(\tau^{k-1} + d^{k-1}) + F(\tau^{k-1}) + H^{k-1} d^{k-1} \|$$
$$\le L_2 \| d^{k-1} \|^2$$

That is,

$$\| F(\tau^{k-1} + d^{k-1}) \| \le L_2 \| d^{k-1} \|^2$$
$$+ \| F(\tau^{k-1}) + H^{k-1} d^{k-1} \| \quad (18)$$

Using (18), (8), (10) and (11), we know

$$dist(\tau^k, X^*) \le c_1 \| F(\tau^k) \|$$
$$= c_1 \| F(\tau^{k-1} + d^{k-1}) \|$$
$$\le c_1 L_2 \| d^{k-1} \|^2 + c_1 \| F(\tau^{k-1}) + H^{k-1} d^{k-1} \|$$
$$\le c_1 L_2 c_2^2 dist(\tau^k, \omega_t^*)^2 + c_1 c_3 dist(\tau^k, \omega_t^*)^2$$
$$\le (c_1 L_2 c_2^2 + c_1 c_3) dist(\tau^k, \omega_t^*)^2$$
$$= c_4 dist(\tau^k, \omega_t^*)^2$$

Where $c_4 = c_1 L_2 c_2^2 + c_1 c_3$.

Last, we prove that the condition of（17）holds. That is, for a positive constant

$\lambda = \min\{\delta/[2(1+2c_2)], 1/(2c_4)\}$，when initial value $\tau^0 \in B_\lambda(\tau^*)$，for any natural number k，we have $\tau^k \in B_{\delta/2}(\tau^*)$.

We prove the above result by mathematical induction.

When $k=0$，from the way λ is chosen, we know $\lambda \le \delta/2$，and then $\tau^0 \in B_{\delta/2}(\tau^*)$. Now assume that $k \ge 0$，$\tau^m \in B_{\delta/2}(\tau^*)$，for $m = 0,1,2,\cdots,k$, We prove in the following $\tau^{k+1} \in B_{\delta/2}(\tau^*)$.

$$\begin{aligned}
&\| \tau^{k+1} - \tau^* \| = \| \tau^k + d^k - \tau^* \| \\
&\le \| \tau^k - \tau^* \| + \| d^k \| \le \| \tau^{k-1} + d^{k-1} - \tau^* \| + \| d^k \| \\
&\le \| \tau^{k-1} - \tau^* \| + \| d^k \| + \| d^{k-1} \| \cdots\cdots \\
&\le \| \tau^0 - \tau^* \| + \sum_{m=0}^{k} \| d^m \| \le \lambda + c_2 \sum_{m=0}^{k} dist(\tau^m, \omega_t^*)
\end{aligned}$$

where, from (10), the last inequality holds. In addition, since $\tau^m \in B_{\delta/2}(\tau^*)$，$m = 0,1,2,\cdots,k$, together with（17）, we have

$$dist(\tau^m, \omega_t^*) \le c_4 dist(\tau^{m-1}, \omega_t^*)^2, m = 0,1,2,\cdots,k$$

Hence

$$\begin{aligned}
&dist(\tau^m, \omega_t^*) \le c_4 dist(\tau^{m-1}, \omega_t^*)^2 \\
&\le c_4 c^2{}_4 dist(\tau^{m-2}, \omega_t^*)^{2^2} \cdots\cdots\cdots \\
&\le c_4 c^2{}_4 \cdots c^{2^{m-1}}{}_4 dist(\tau^0, \omega_t^*)^{2^m} \\
&\le c_4{}^{2^m - 1} \| \tau^0 - \tau^* \|^{2^m} \le c_4{}^{2^m-1} \lambda^{2^m}
\end{aligned}$$

From the above formula, and the way λ is chosen, we know that $\lambda \le 1/(2c_4)$，$\lambda \le \delta/[2(1+2c_2)]$,

And

$$\begin{aligned}
&\| \tau^{k+1} - \tau^* \| \le \lambda + c_2 \sum_{m=0}^{k} dist(\tau^m, \omega_t^*) \\
&\le \lambda + c_2 \sum_{m=0}^{k} c_4{}^{2^m-1} \lambda^{2^m} \quad \le \lambda + c_2 \lambda \sum_{m=0}^{k} c_4{}^{2^m-1} \lambda^{2^m-1} \\
&\le \lambda + c_2 \lambda \sum_{m=0}^{k} \left(\frac{1}{2}\right)^{2^m-1} \le \lambda + c_2 \lambda \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \\
&\le (1+2c_2)\lambda \le \delta/2
\end{aligned}$$

Now Theorem 3.1 is proved. □

NOTE: Theorem 3.1shows that the given smooth algorithm has the property of quadratic convergence without the condition of existence of a non-degenerate solution. This is a new result.

V. Conclusions

In this paper, we propose an algorithm for solving the management equilibrium model. Under without the requirement of nondegenerate solution, we also show that the algorithm is quadratic convergence based on error bound estimation instead of the nonsingular assumption just as was done in [5,6]. This conclusion can be viewed as extension of previously known result in [5, 6]. How to use the algorithm to solve the practical management based on the computer, this is a topic for future research.

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