Output Feedback Controller Synthesis for Discrete-Time Nonlinear Systems

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***Abstract*—This paper presents a computational approach for solving optimal control problem for a class of nonlinear discrete-time systems. We focus on problem in which a pre-specified N local subsystems are given to describe the studied systems. For such problem, we derive an output feedback controller and a cost function such that the resulting closed-loop system is asymptotically stable and the closed loop cost function is minimized. The main results are demonstrated numerically through the implementation of the proposed algorithm for solving the optimal control problem of a mechanical system.**

Keywords—nonlinear systems; discrete-time systems; optimal control; output feedback control

I. Introduction

The optimal control of discrete-time nonlinear systems remains an important open control problem in control engineering and continue to attract significant attention of control research [1-4]. For the static output feedback, the reader is referred to [5-9]. It is well known that the optimal control of linear systems with respect to a quadratic cost function can be achieved by solving the Riccati equation [10].

However, when studying the nonlinear discrete-time systems we often need to solve either nonlinear partial difference or differential Hamilton-Jacobi-Bellman equation [11,12], which is generally a difficult task. Despite recent advances [13-17], some of the developed techniques have limited applicability because of the strong conditions imposed on the system. Other implemented solutions are often partial and of significant complexity because of the need to find an accurate model of the system. In practice, an accurate complete model of the studied system isn't usually available. The full model information cannot be available at the time of design or it might change or the system can have different operating conditions. For all the previous reasons and even for the reason of simplifying the tuning and the maintenance of the system we need to rely on local model control based on local modeling. Motivating by what discussed above we will investigate the best closed loop performance that is achievable by an output feedback controller based on multi-model approach [18-20].

The rest of the paper is organized as follows: Section 2 provides the description of the studied systems and problem statements. The proposed strategy of the optimal output feedback control of the discrete time nonlinear studied systems is presented in section 3. Main results are derived and summarized by an efficient algorithm. Then, the validity of the proposed approach is illustrated by simulation results in Section 4. Concluding remarks are given in Section 5.

II. Problem Statements

Let's consider in this study a class of nonlinear and uncertain discrete-time systems described as:

$$(S)\begin{cases} x(k+1) = f\left(x(k), u(k), \theta(k)\right) \\ y(k) = h\left(x(k), \theta(k)\right) \\ \theta(k) \in ID \end{cases} \tag{1}$$

where $x(k) \in \Re^n$ is the state vector, $u(k) \in \Re^m$ is the input vector and $y(k) \in \Re^p$ is the output vector. The functions $f(.)$ and $h(.)$ depend on a vector of parameters $\theta(k)$ which is considered unknown, but evolving in a convex domain ID.

In the literature, various approaches [15,16] like identification, linearization or convex polytopic transformation can be used to determine the multi-model description of a complex system.

In this paper, we assume that the nonlinear mathematical model of the studied system is known. By linearization around its several operating points $(u_{i0}, x_{i0}),\, i = 1 \ldots N$, different and simpler local models are obtained. So the complex studied system described initially by a nonlinear mathematical model (1) can be then described by a library of N local linear model characterized by the following state space equations:

$$(M_i)\begin{cases} x_i(k+1) = A_i x_i(k) + B_i u_i(k) \\ y_i(k) = C_i x_i(k) \\ i = 1, \ldots, N \end{cases} \tag{2}$$

where N *is* the number of local models, $x_i(k) \in \Re^n$, $y_i(k) \in \Re^p$ and $u_i(k)$ are respectively the state vector, the output vector and the control input vector of the *i-th* submodel noted M_i.

The state space matrices A_i, B_i, C_i are constant of appropriate dimensions to be determined.

$$A_i = \frac{\partial f}{\partial x}\bigg|_{(u_{i0}, x_{i0})},\ B_i = \frac{\partial f}{\partial u}\bigg|_{(u_{i0}, x_{i0})},\ C_i = \frac{\partial h}{\partial x}\bigg|_{(u_{i0}, x_{i0})} \quad (3)$$

and let's note:

$$\begin{cases} x_i(k) = x(k) - x_{i0} \\ u_i(k) = u(k) - u_{i0} \\ i = 1, \dots, N \end{cases} \quad (4)$$

Each model of the library, involving N sub-models, contributes to the process description with a degree of trust measured by a validity coefficient. The validity appears to be of great importance if realizing their influence on the performances of the global control law.

Indeed, the use of the validity coefficients is a convenient mean to experiment with sub-collection of systems and is also useful to put more emphasis on the performances of some particular instances of parameter values. In the literature several methods were proposed for the estimation of these validities. In this paper the approach proposed in [19] is considered for validities computing. Because the implementation of linear controllers is straightforward and cost effective, the multi-model approach was also proposed in the following section as a solution for the control and analysis of the nonlinear studied discrete systems.

III. Optimal Output Feedback Control for Discrete-Time Nonlinear Systems

In this section, our objective is to design an output feedback controller for the studied system and a cost function such that the resulting closed-loop system is asymptotically stable and the closed loop cost function is minimized.

Assume for each isolated subsystem M_i, a local controller is designed.

$$u_i(k) = -F_i y_i(k) \quad (5)$$

where $F_i \in \Re^{m\times p}$ is the control gain matrix to be determined by minimizing the proposed quadratic function:

$$J_i = \sum_{k=0}^{\infty}\left(x_i^T(k)Q_i x_i(k) + u_i^T(k)R_i u_i(k)\right) \quad (6)$$

where $Q_i = Q_i^T \geq 0 \in \Re^{n\times n}$ and R_i are the state and input weighting matrices.

Applying controller (5) to the system (2) results in the closed-loop system:

$$x_i(k+1) = (A_i - B_i F_i C_i)x_i(k) \quad (7)$$

The performance index associated with the studied system (1) is then the following quadratic function

$$\begin{aligned} J &= \sum_{i=1}^{N}\mu_i J_i \\ &= \sum_{i=1}^{N}\mu_i \sum_{k=0}^{\infty} x_i^T(k)(Q_i + C_i^T F_i^T R_i F_i C_i)x_i(k) \end{aligned} \quad (8)$$

where μ_i, $i = 1, \dots, N$ are the validity coefficients of the proposed multi-model description.

Using the solution of the recurrent equation (7), one can write:

$$x_i(k) = (A_i - B_i F_i C_i)^K x_i(0) \quad (9)$$

and substituting (9) in (8), the global performance index (8) can be rewritten:

$$J = \sum_{i=1}^{N}\mu_i \sum_{k=0}^{\infty} x_i^T(0) * \left[\left((A_i - B_i F_i C_i)^K\right)^T\left(Q_i + C_i^T F_i^T R_i F_i C_i\right)(A_i - B_i F_i C_i)^K\right]x_i(0)$$

and presented in a simplified expression:

$$J = \sum_{i=0}^{N}\mu_i x_{i0}^T P_i x_{i0} \quad (10)$$

where

$$P_i = \sum_{k=0}^{\infty}\left[\left((A_i - B_i F_i C_i)^K\right)^T\left(Q_i + C_i^T F_i^T R_i F_i C_i\right)(A_i - B_i F_i C_i)^K\right] \quad (11)$$

are symmetric positive definite matrices, solutions of the following Lyapunov equations:

$$\begin{aligned} &\left(A_i - B_i F_i C_i\right)^T P_i \left(A_i - B_i F_i C_i\right) - P_i + \\ &Q_i + C_i^T F_i^T R_i F_i C_i = 0\ ; \quad i = 1, \dots, N \end{aligned} \quad (12)$$

The dependency of the optimal solution on the initial condition can be removed when considering the average value function $E(.)$ such that:

$$E\left(x_{i0}^T P_i x_{i0}\right) = I_n \quad (13)$$

Based on equation (13), the corresponding closed-loop cost function will be written as follows:

$$\bar{J} = \sum_{i=1}^{N}\mu_i trace\left\{P_i\right\} \quad (14)$$

A. Main Results

In order to derive the necessary conditions of optimal gain matrices of the feedback control, the optimization problem formulated by (11) is reduced to the minimization of the following Lagrangian:

$$\zeta(F_i, P_i, S_i) = \sum_{i=1}^{N} \mu_i \, trace\{P_i\} + \sum_{i=1}^{N} \mu_i \, trace\left\{\Gamma_i^T\left[\left((A_i - B_i F_i C_i)^T P_i (A_i - B_i F_i C_i) - P_i + Q_i + C_i^T F_i^T R_i F_i C_i\right)\right]\right\} \quad (15)$$

where $\Gamma_i \in \Re^{n \times n}, \Gamma_i = \Gamma_i^T \geq 0, i = 1, \dots, N$ selected to be symmetric positive definite matrices are Lagrange multipliers.

By using the gradient matrix operations [20,21], the necessary conditions for F_i, P_i and Γ_i, to be optimal are given by

$$\begin{cases} \dfrac{\partial\zeta(F_i,P_i,\Gamma_i)}{\partial F_i} = \\ -2\sum_{i=1}^{N} \mu_i \left[B_i^T P_i A_i \Gamma_i C_i^T - B_i^T P_i B_i F_i C_i \Gamma_i C_i^T - R_i F_i C_i \Gamma_i C_i^T \right] = 0 \\ \dfrac{\partial\zeta(F_i,P_i,\Gamma_i)}{\partial P_i} = \\ \sum_{i=1}^{N} \mu_i \left[(A_i - B_i F_i C_i)\Gamma_i (A_i - B_i F_i C_i)^T - \Gamma_i + I_n \right] = 0 \\ \dfrac{\partial\zeta(F_i,P_i,\Gamma_i)}{\partial \Gamma_i} = \\ \sum_{i=1}^{N} \mu_i \left[(A_i - B_i F_i C_i)^T P_i (A_i - B_i F_i C_i) - P_i + Q_i + C_i^T F_i^T R_i F C_i \right] = 0 \\ i = 1, \dots, N \end{cases} \quad (16)$$

Solving the first equation in (16), one obtains the optimal control gain matrix F_i of the local model M_i:

$$F_i = \left(B_i^T P_i B_i + R_i\right)^{-1} \left(B_i^T P_i A_i \Gamma_i C_i^T\right)\left(C_i \Gamma_i C_i^T\right)^{-1} \quad (17)$$

and from the two others equations we can determine the matrices Γ_i and all the matrices P_i solutions of the Lyapunov equations (12). Indeed, based on (16), Γ_i and P_i are also the solutions of the following equations:

$$\begin{cases} G_2(F_i, \Gamma_i) = 0 \\ G_3(F_i, P_i) = 0 \end{cases}$$

$$G_2(F_i, \Gamma_i) = (A_i - B_i F_i C_i)\Gamma_i (A_i - B_i F_i C_i)^T - \Gamma_i + I_n \quad (18)$$

$$G_3(F_i, P_i) = (A_i - B_i F_i C_i)^T P_i (A_i - B_i F_i C_i) - P_i \; Q_i + C_i^T F_i^T R_i F_i C_i \quad (19)$$

To solve instantly the three equations (17), (18) and (19), and to calculate all the introduced matrices F_i, P_i and Γ_i, we propose an iterative algorithm which can be summarized in the following way:

Algorithm (A)

Step1 : Initialize $n = 1$

Select $Q_i \geq 0, R_i > 0$ and an initial matrix F_{i0} as initial starting value such that $A_i - B_i F_{i0} C_i$ is a stable for each local model.

Step 2 : n[th] iteration

- calculate F_{in} (17)
- solve $G_2(F_{in}, P_{in}) = 0$ and calculate Γ_{in}.
- solve $G_3(F_{in}, P_{in}) = 0$; and get the matrix P_{in}.
- calculate

$$F_{i(n+1)} = \left(B_i^T P_{in} B_i + R_i\right)^{-1}\left(B_i^T P_{in} A_i \Gamma_{in} C_i^T\right) \times \left(C_i \Gamma_{in} C_i^T\right)^{-1}$$

Step 3 : incrementation

repeat step 2 until verifying $\left\| P_{in} - P_{i(n-1)} \right\| \leq \varepsilon$

End

ε is a prescribed small number used to check the convergence of the algorithm.

B. The Optimal Controller Design

Given the predetermined matrices F_i, the system (1) can be controlled in an optimal manner by the following control policy $u(k)$, which guarantees the minimization of the infinite horizon cost function (8).

$$u(k) = -\sum_{j=1}^{N} \mu_j F_j \; y(k) \quad (20)$$

then the closed-loop system (1) admits the realization:

$$\begin{cases} x(k+1) = f\left(x(k), u(k), \theta(k)\right) \\ y(k) = h\left(x(k), u(k)\right) \\ u(k) = -Fy(k) \end{cases}$$

where

$$F = \sum_{i=1}^{N} \mu_i F_i \;, \; F \in \Re^{p \times m} \quad (21)$$

C. Stability Analysis

In order to prove the asymptotic stability of the controlled system, let's consider $V(x_i(k))$ the Lyapunov function defined by the following quadratic form:

$$V(x_i(k)) = x_i^T(k) P_i x_i(k) \tag{22}$$

where $P_i \in \Re^{n\times n}$ are the symmetric positive definite matrices solution of the equation (12) and (16).

The stability of the controlled system (7) is ensured if the difference of Lyapunov function (22) along the trajectory of (7) is negative definite.

One has

$$\begin{aligned}\Delta V(k) &= V\left(x_i(k+1)\right) - V\left(x_i(k)\right)\\ &= x_i^T(k+1)P_i x_i(k+1) - x_i^T(k)P_i x_i(k)\\ &= x_i^T(k)\left[(A_i - B_iF_iC_i)^T P_i (A_i - B_iF_iC_i) - P_i\right]x_i(k)\end{aligned} \tag{23}$$

Using the third equation of system (16), (19) becomes:

$$\Delta V(k) = -x_i^T(k)\left[Q + C_i^T F_i^T R_i F_i C_i\right]x_i(k) \tag{24}$$

According to the properties of matrices Q_i and R_i, the matrix $Q_i + C_i^T F_i^T R F_i C$ is symmetric positive definite. The variation of the quadratic Lyapunov function, expressed by (24), is then negative defined and the controlled system is then asymptotically stable.

IV. Application to a Mechanical System

In order to demonstrate the effectiveness and merits of the proposed optimal output feedback controller over the existing results, the following mechanical system described by a spring damper mass M is considered:

$$M\ddot{x}(t) + c_1\dot{x}(t) + c_2 x(t) = \left(1 + c_3\dot{x}^3(t)\right)u(t) \tag{25}$$

where:

- $M = 1Kg$ is the mass of the system,
- $c_1 = 1$, $c_2 = 1.155$ and $c_3 = 0.13$ are constants,
- $u(t)$ is the exerted force for the spring,
- $\dot{x}^3(t)$ is the nonlinear term.

and rewritten in the following state space equations:

$$\begin{cases}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-\frac{c_1}{M} & -\frac{c_2}{M}\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}\frac{1}{M}\left(1 + c_3x_1^3(t)\right)\\ 0\end{bmatrix}u(t)\\ \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}x_1(t)\\ 2x_1(t) + x_2(t)\end{bmatrix}\end{cases} \tag{26}$$

where $x_1(t)$ is the velocity of the mass and $x_2(t)$ the position of the same mass.

By using an appropriate discretization method and a suitable sampling period $T = 0.05s$, it comes the discrete-time state space equations:

$$\begin{cases}\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix} = \begin{bmatrix}1 - T\frac{c_1}{M} & -T\frac{c_2}{M}\\ T & 1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}\\ \qquad + \begin{bmatrix}\frac{T}{M}\left(1 + c_3x_1^3(k)\right)\\ 0\end{bmatrix}u(k)\\ \begin{bmatrix}y_1(k)\\ y_2(k)\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}\end{cases}$$

we pointed out that the nonlinearity of the system is considered as uncertainty and the term of linearities depend on $x_1(k)$ which is assumed to vary in the range $[-1.5 \quad 1.5]$.

According to section 3, and based on the multimodel approach the nonlinear dynamical system (26) can be described by:

$$\begin{cases}x(k+1) = \sum_{i=1}^{2}\mu_i\left(x_1(k)\right)\left(A_ix(k) + B_iu(k)\right)\\ y(k) = \sum_{i=1}^{2}\mu_i\left(x_1(k)\right)C_ix(k)\end{cases} \tag{27}$$

where

$$x(k) = \begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix} \text{ and } A_1 = A_2 = \begin{bmatrix}0.9 & -0.1155\\ 0.1 & 1\end{bmatrix},$$

$$B_1 = \begin{bmatrix}0.1439\\ 0\end{bmatrix}, B_2 = \begin{bmatrix}0.0561\\ 0\end{bmatrix}, C_1 = C_2 = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}.$$

The validity coefficients of this system are expressed as follows:

$$\mu_1(x_1(k)) = 0.5 + \frac{x_1^3(k)}{6.75}$$

$$\mu_2(x_1(k)) = 1 - \mu_1(x_1(k))$$

Using the proposed iterative algorithm the following results are derived:

- The quadratic criterion:

$$J_1 = 0.7727,\ J_2 = 0.9708$$

- The symmetric positive definite matrices:

$$P_1 = \begin{bmatrix} 0.1412 & 0.1148 \\ 0.1148 & 0.6315 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.2735 & 0.1806 \\ 0.1806 & 0.6973 \end{bmatrix}$$

- The symmetric positive definite matrices of Lagrange multipliers:

$$\Gamma_1 = \begin{bmatrix} 5.9552 & -5.2978 \\ -5.2978 & 7.0525 \end{bmatrix},\ \Gamma_2 = \begin{bmatrix} 6.9638 & -5.3482 \\ -5.3482 & 9.8512 \end{bmatrix}$$

and all the gain matrices of the proposed optimal control are calculated:

$$F_1 = \begin{bmatrix} -1.0621 & 1.7891 \end{bmatrix},\ F_2 = \begin{bmatrix} -0.3236 & 1.4268 \end{bmatrix}$$

To show the effectiveness of the proposed optimal output feedback control we have carried out some simulations shown from figure 1 to 2. It appears from figure1 a satisfactory stabilization of the state variables of the controlled discrete-time studied system. The figure 2 illustrates the evolution of the proposed optimal output feedback control law. Indeed, its high performances shows the aptitude of the proposed **Algorithm (A)** to be implemented and to give interesting results for the output feedback control of a large class of nonlinear discrete-time systems.

V. CONCLUSION

An iterative algorithm is proposed to derive all the gain matrices of the designed optimal feedback control. The nonlinear discrete-time studied system is first represented by a multi local linear models. Then, an output feedback controller based on the multimodel control approach and minimizing a quadratic criterion is derived assuring the asymptotic stability of the controlled system. The gradient resolution of the Lagrangian functions and the iterative algorithm allowed the calculus of all the gain matrices. An illustrative example of a mechanical system is considered and the simulation results show the effectiveness of the proposed control strategy.

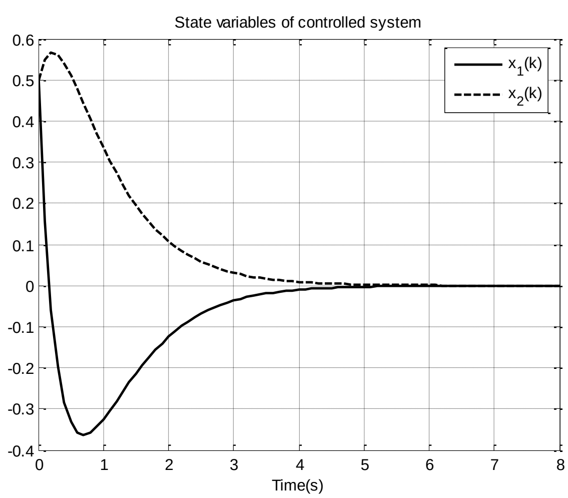

Fig. 1. State trajectories subject to the proposed control

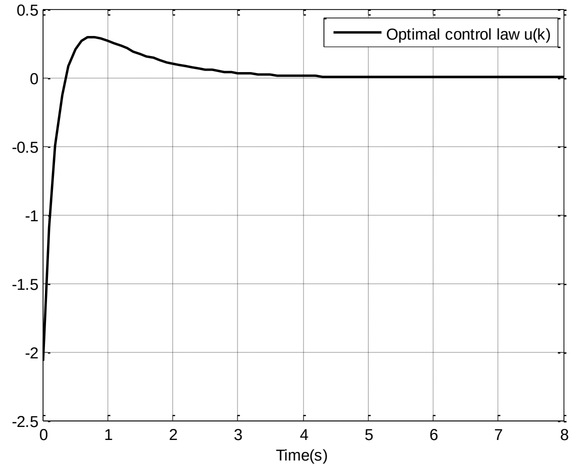

Fig. 2. The proposed optimal control law

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